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LOWER LIMIT TO THE STRENGTH OF SURFACE FORCES IN THE CASE OF PLANE STRAIN OF AN IDEAL RIGID-PLASTIC MEDIUM

A. E. Alekseev

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A lower limit to the strength of surface forces based on the use of a statically permissible stress field follows from the extremum theorems of an ideal rigid-plastic medium [1]. It is also known that the stress field in a rigid-plastic medium with a convex plasticity condition is unique in those zones in which the deformation rates are different from zero [2]. It is shown in this paper that there exists for the class of problems in which a functional corresponding to the lower limit of the strength of the external surface forces is nonidentically equal to a constant on a set of statically permissible stress fields a stress field which yields a maximum of this functional.

1. Let Ω be a region with a piecewise-continuous boundary S on the (x, y) plane, and let $\mathrm{mes}(\Omega) < \infty$. A stress field $(\sigma_x, \sigma_y, \tau)$ which is continuous and continuously differentiable satisfies the equilibrium conditions in Ω

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau}{\partial y} + f_x = 0, \quad \frac{\partial\tau}{\partial x} + \frac{\partial\sigma_y}{\partial y} + f_y = 0, \tag{1.1}$$

and the boundary conditions on part of the boundary S_σ

$$\begin{aligned} &\sigma_n = \sigma_x n_x^2 + \sigma_y n_y^2 + 2\tau n_x n_y = g(S),\\ &\tau_n = (\sigma_y - \sigma_x) n_x n_y + \tau (n_x^2 - n_y^2) = h(S) \end{aligned} \tag{1.2}$$

and does not violate the plasticity condition in $\overline{\Omega} = \Omega + S$,

$$\frac{1}{4}(\sigma_x - \sigma_y)^2 + \tau^2 \leqslant \tau_s^2 \tag{1.3}$$

is called statically permissible.

A velocity field (u, v) which satisfies the incompressibility condition in Ω

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.4}$$

and the boundary conditions on the part of the boundary $S_u = S - S_\sigma$

$$u = u_0(S), \ v = v_0(S) \tag{1.5}$$

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is called kinematically possible.

In (1.1)–(1.3) and (1.5) f_x, f_y, h, g, u_0, v_0 are specified functions, n_x and n_y are the cosines of the outer normals to S, and τ_s is the yield point for pure shear.

The coupling equation between the velocities and stresses for an ideal rigid-plastic medium with the Mises plasticity condition is of the form

$$\frac{\sigma_x - \sigma_y}{2\tau} = \frac{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}}{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}}. \tag{1.6}$$

It follows from the extremum theorems for an ideal rigid-plastic body [1] that

(1.7) $$I_1(\sigma_x^*, \sigma_y^*, \tau^*) \geqslant \sup_{(\sigma_x, \sigma_y, \tau)\in G} I_1(\sigma_x, \sigma_y, \tau), \tag{1.7}$$

where

$$I_1(\sigma_x, \sigma_y, \tau) = \int_{S_u} [(\sigma_x u_0 + \tau v_0) n_x + (\tau u_0 + \sigma_y v_0) n_y] dS \tag{1.8}$$

is a linear functional on the set G of statically permissible stress fields and σ_x^*, σ_y^*, τ are the stresses corresponding to the solution of the problem (1.1)–(1.6).

Let (u, v) be any continuous and continuously differentiable velocity field in Ω which is kinematically possible. Then using the Gauss–Ostrogradskii formula and the incompressibility condition (1.4), one can reduce the functional (1.8) to the form

$$\begin{aligned} I_1(\sigma_x, \sigma_y, \tau) &= \int_\Omega \left(\frac{\partial u}{\partial x}(\sigma_x - \sigma_y) + \tau\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) d\Omega \\ &- \int_{S_\sigma} [g(S) v_n + h(S) v_t]\, dS - \int_\Omega (f_x u + f_y v)\, d\Omega \equiv I_2(\sigma_x, \sigma_y, \tau), \end{aligned} \tag{1.9}$$

where v_n, v_t are the normal and tangential velocity components on the surface S.

2. Let us assume that the set G is not empty and the functional I_2 is not identically equal to a constant on G. Let $(\bar\sigma_x, \bar\sigma_y, \bar\tau)$ be some statically permissible stress field. Let $\varphi \in C^3(\Omega)$ so that

$$\frac{1}{4}\left(\frac{\bar\sigma_x}{\tau_s} - \frac{\bar\sigma_y}{\tau_s} + \frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial x^2}\right)^2 + \left(\frac{\bar\tau}{\tau_s} - \frac{\partial^2\varphi}{\partial x \partial y}\right)^2 \leqslant 1 \text{ in } \bar\Omega; \tag{2.1}$$

$$\frac{\partial^2\varphi}{\partial S^2} = 0, \quad \frac{\partial^2\varphi}{\partial n \partial S} = 0 \text{ on } S_\sigma. \tag{2.2}$$

Then any stress field $(\sigma_x, \sigma_y, \tau)$ satisfying the relationships

$$\frac{\sigma_x}{\tau_s} = \frac{\bar\sigma_x}{\tau_s} + \frac{\partial^2\varphi}{\partial y^2}, \quad \frac{\sigma_y}{\tau_s} = \frac{\bar\sigma_y}{\tau_s} + \frac{\partial^2\varphi}{\partial x^2}, \quad \frac{\tau}{\tau_s} = \frac{\bar\tau}{\tau_s} - \frac{\partial^2\varphi}{\partial x \partial y}, \tag{2.3}$$

is statically permissible.

Using (2.3), we will write the functional (1.9) in the form

$$I_2(\sigma_x, \sigma_y, \tau) = I_2(\bar\sigma_x, \bar\sigma_y, \tau) + I_0(\varphi), \tag{2.4}$$

where

$$I_0(\varphi) = \int_\Omega \left[-\frac{\partial u}{\partial x}\left(\frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial x^2}\right) - \frac{\partial^2\varphi}{\partial x \partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] d\Omega \tag{2.5}$$

is a linear functional on the set of functions $\varphi \in C^3(\Omega)$ satisfying the conditions (2.2).

The function φ is determined according to (2.3) from the given stress field $(\sigma_x, \sigma_y, \tau)$ to an accuracy out to a linear function; therefore, in order to establish a one-to-one correspondence between the set of functions φ satisfying (2.1) and (2.2) and the set G of statically permissible stress fields, we set

$$\int_\Omega \varphi\, d\Omega = 0, \quad \int_\Omega \frac{\partial\varphi}{\partial x}\, d\Omega = 0, \quad \int_\Omega \frac{\partial\varphi}{\partial y}\, d\Omega = 0. \tag{2.6}$$

Let M be a set of functions $\varphi \in C^3(\Omega)$ satisfying (2.2) and (2.6), and let M_1 be a subset of functions from M for which the inequality (2.1) is valid. Then using (1.9) and (2.5), it is possible to represent the inequality (1.7) in the form

$$I_2(\sigma_x^*, \sigma_y^*, \tau^*) \geqslant I_2(\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}) + \sup_{\varphi \in M_1} I_0(\varphi). \tag{2.7}$$

Let us denote by H the Hilbert space corresponding to M with the scalar product

$$(\varphi, \psi) = \int_\Omega \left[\frac{\partial^2 \varphi}{\partial x \partial y} \frac{\partial^2 \psi}{\partial x \partial y} + \frac{1}{4} \left(\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} \right) \left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \right) \right] d\Omega$$

and the norm

$$\|\varphi\|_H^2 = (\varphi, \varphi). \tag{2.8}$$

We will show that $\varphi = 0$ follows from $(\varphi, \varphi) = 0$. The remaining axioms of a scalar product are satisfied in an obvious way. Let $(\varphi, \varphi) = 0$; then we have from (2.7) and (2.8)

$$\varphi = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 (x^2 + y^2).$$

If $S_\sigma \neq \phi$, then it follows from (2.2) and (2.6) that $\alpha_i = 0$ $(i = 0, \dots, 3)$ and $\varphi \equiv 0$. We note that the second of the conditions (2.2) is satisfied for any α_i.

Let us consider a set $N \subset H$ such that almost everywhere in Ω

$$\frac{1}{4} \left(\frac{\bar{\sigma}_x}{\tau_s} - \frac{\bar{\sigma}_y}{\tau_s} + \frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \varphi}{\partial x^2} \right)^2 + \left(\frac{\bar{\tau}}{\tau_s} - \frac{\partial^2 \varphi}{\partial x \partial y} \right)^2 \leqslant 1, \quad \varphi \in N. \tag{2.9}$$

It follows from (1.3) and (2.9) that

$$\|\varphi\|_H^2 = \int_\Omega \left[\frac{1}{4} \left(\frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \varphi}{\partial x^2} \right)^2 + \left(\frac{\partial^2 \varphi}{\partial x \partial y} \right)^2 \right] d\Omega \leqslant 2 \left\{ \frac{1}{\tau_s^2} \int_\Omega \left[\frac{1}{4} (\bar{\sigma}_x - \bar{\sigma}_y)^2 + \bar{\tau}^2 \right] d\Omega \right.$$
$$\left. + \int_\Omega \left[\frac{1}{4} \left(\frac{\bar{\sigma}_x}{\tau_s} - \frac{\bar{\sigma}_y}{\tau_s} + \frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \varphi}{\partial x^2} \right)^2 + \left(\frac{\bar{\tau}}{\tau_s} - \frac{\partial^2 \varphi}{\partial x \partial y} \right)^2 \right] d\Omega \right\}$$
$$\leqslant 4 \operatorname{mes}(\Omega) < \infty, \quad \varphi \in N.$$

Consequently, N is bounded.

We will show that N is a strongly convex set, i.e., there exists a constant $\gamma > 0$ for which any function $\varphi = (\varphi_1 + \varphi_2)/2 + \psi \in N$ if $\varphi_1, \varphi_2 \in N$ and $\|\psi\|_H \leq \gamma \|\varphi_1 - \varphi_2\|_H$. We will denote

$$L\varphi = \left[\frac{1}{4} \left(\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} \right)^2 + \left(\frac{\partial^2 \varphi}{\partial x \partial y} \right)^2 \right]^{1/2},$$
$$\bar{L}\varphi = \left[\frac{1}{4} \left(\frac{\bar{\sigma}_x}{\tau_s} - \frac{\bar{\sigma}_y}{\tau_s} + \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} \right)^2 + \left(\frac{\bar{\tau}}{\tau_s} - \frac{\partial^2 \varphi}{\partial x \partial y} \right)^2 \right]^{1/2}. \tag{2.10}$$

Omitting the obvious calculations, we have

$$\left(\bar{L} \left(\frac{\varphi_1 + \varphi_2}{2} + \psi \right) \right)^2 \leqslant \frac{1}{2} (\bar{L}\varphi_1)^2 + \frac{1}{2} (\bar{L}\varphi_2)^2 - \frac{1}{4} (L(\varphi_1 - \varphi_2))^2$$
$$+ (L\psi)^2 + 2L\psi \sqrt{\frac{1}{2} (\bar{L}\varphi_1)^2 + \frac{1}{2} (\bar{L}\varphi_2)^2 - \frac{1}{4} (L(\varphi_1 - \varphi_2))^2}$$
$$= \left(L\psi + \sqrt{\frac{1}{2} (\bar{L}\varphi_1)^2 + \frac{1}{2} (\bar{L}\varphi_2)^2 - \frac{1}{4} (L(\varphi_1 - \varphi_2))^2} \right)^2.$$

Since $\varphi_1, \varphi_2 \in N$, we obtain from this

$$\left(\bar{L} \left(\frac{\varphi_1 + \varphi_2}{2} + \psi \right) \right)^2 \leqslant \left(L\psi + \sqrt{1 - \frac{1}{4} (L(\varphi_1 - \varphi_2))^2} \right)^2 \leqslant \left(1 - \frac{1}{8} (L(\varphi_1 - \varphi_2))^2 + L\psi \right)^2.$$

Let ψ be an arbitrary function which satisfies the condition

$$L\psi \leqslant \frac{1}{8} (L(\varphi_1 - \varphi_2))^2, \tag{2.11}$$

then

$$\left(\bar{L} \left(\frac{\varphi_1 + \varphi_2}{2} + \psi \right) \right)^2 \leqslant 1$$

and the function

$$\frac{\varphi_1+\varphi_2}{2}+\psi\in N.$$

It follows from (2.8) and (2.10) that

$$\|\varphi\|_H^2=\int_\Omega (L\varphi)^2\,d\Omega. \tag{2.12}$$

It is obvious that for $\varphi_1, \varphi_2 \in N$

$$(1/2)L(\varphi_1-\varphi_2)\leqslant 1 \tag{2.13}$$

is valid. From (2.11)–(2.13) we have the chain of inequalities

$$\|\psi\|_H^2=\int_\Omega (L\psi)^2\,d\Omega\leqslant\frac{1}{64}\int_\Omega (L(\varphi_1-\varphi_2))^4\,d\Omega\leqslant\frac{1}{16}\int_\Omega (L(\varphi_1-\varphi_2))^2\,d\Omega=\frac{1}{16}\|\varphi_1-\varphi_2\|_H^2,$$

from which it follows that it is sufficient to set $\gamma = 1/4$.

Following the results of [3], one can show that N is closed. Thus N is a bounded strongly convex set.

Let (u, v) be some kinematically possible velocity field which is continuous and continuously differentiable in Ω. Then we have from (2.5) and the Cauchy–Bunyakovskii inequality

$$I_0(\varphi)\leqslant C\|\varphi\|_H,$$

where

$$C=\left(\int_\Omega\left(4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right)d\Omega\right)^{1/2},$$

and, consequently, $I_0(\varphi)$ is a linear bounded functional specified on the set M, which is dense in H. It follows from the theorems of functional analysis that it is possible in this case to continue $I_0(\varphi)$ in a unique fashion onto the entire set H. The continued functional $I_0(\varphi)$ is continuous in H.

The existence of a unique element $\varphi^* \in N$ such that

$$I_0(\varphi^*)=\sup_{\varphi\in N} I_0(\varphi) \tag{2.14}$$

follows from the next statement.

Let H be a Hilbert space, $l\varphi$ a linear continuous functional, and $V \subset H$ a strongly convex bounded closed set with boundary Q. Then there exists a unique element $\varphi \in Q$ for which

$$l\varphi=\sup_{\psi\in V}(\inf)\,l\psi.$$

The proof of this statement is similar to the proof of the theorem on the minimum of a quadratic functional with one-sided limits [3, 4].

Let $\{\psi_n\} \in V$ be a sequence such that

$$\lim_{n\to\infty} l\psi_n=\sup_{\psi\in V} l\psi. \tag{2.15}$$

Since V is bounded,

$$\|\psi_n\|_H\leqslant K<\infty$$

and it is possible to derive a sequence $\{\psi_{nk}\}$ such that

$$\lim_{n\to\infty} l\psi_{nk}=l\chi,\quad \chi\in H. \tag{2.16}$$

It follows from the closed nature and convexity of V that V is weakly closed. Then $\chi \in V$, and setting $\varphi = \chi$, we have

$$l\varphi=\sup_{\psi\in V} l\psi,\quad \varphi\in V$$

from (2.15) and (2.16).

We will show that $\varphi \in Q$. Let us assume the opposite. Then there exists a $\delta > 0$ such that $V_\delta \subset V$ and

$V_\delta = (\psi \,|\, \|\varphi - \psi\| < \delta)$. According to the Riess theorem on the form of a linear continuous functional in a Hilbert space, we have

$$l\varphi = (\varphi, \varphi_0),\ \varphi_0 \in H.$$

Let us consider an element $\psi_1 \in H$ such that

$$\psi_1 = \varphi + \frac{\delta}{2}\frac{\varphi_0}{\|\varphi_0\|_H}.$$

It is evident that $\psi_1 \in V_\delta$ and in addition

$$l\psi_1 = (\psi_1, \varphi_0) = (\varphi, \varphi_0) + \frac{\delta}{2}\|\varphi_0\| > l\varphi = \sup_{\psi \in V} l\psi.$$

We obtain a contradiction. Consequently, $\varphi \in Q$. Since $\varphi \in Q$, the uniqueness follows from the linearity of the functional and the strong convexity of V.

It is similarly proven that there exists a unique element $\varphi \in Q$ for which

$$l\varphi = \inf_{\psi \in V} l\psi.$$

3. The maximum of the functional $I_0(\varphi)$ is determined on the set $N \subset H$. Therefore, the problem of in what sense the stresses (2.3), which correspond to the element $\varphi \in N$, satisfy the equilibrium equations (1.1) and the boundary conditions (1.3) is of interest.

It follows from (2.8) that if $\varphi \in H$, the derivatives $\partial^2\varphi/\partial x\partial y$ and $(\partial^2\varphi/\partial y^2 - \partial^2\varphi/\partial x^2)$ are integrable with a square in Ω. Consequently, the stresses

$$\begin{gathered}\sigma_x - \sigma_y = \bar{\sigma}_x - \bar{\sigma}_y + \tau_s(\partial^2\varphi/\partial y^2 - \partial^2\varphi/\partial x^2),\\ \tau = \bar{\tau} - \tau_s\partial^2\varphi/\partial x\partial y,\end{gathered} \tag{3.1}$$

where

$$(\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}) \in G,$$

are also integrable with a square in Ω.

The space H is a supplement of the set M with respect to the norm (2.8); consequently, there exists a sequence $\{\varphi_n\} \in M$ such that

$$\|\varphi - \varphi_n\|_H \to 0,\ n \to \infty. \tag{3.2}$$

In this connection the identity

$$\int_\Omega \left[\left(\frac{\partial^2\varphi_n}{\partial y^2} - \frac{\partial^2\varphi_n}{\partial x^2}\right)\frac{\partial \delta u}{\partial x} - \frac{\partial^2\varphi_n}{\partial x\partial y}\left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x}\right)\right] d\Omega \equiv 0,$$

where $\delta u = u_2 - u_1$, $\delta v = v_2 - v_1$, and (u_1, v_1) and (u_2, v_2) are arbitrary continuous and continuously differentiable kinematically possible velocity fields, is valid for each φ_n. Consequently,

$$\begin{aligned}&\left|\int_\Omega \left(\left(\frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial x^2}\right)\frac{\partial \delta u}{\partial x} - \frac{\partial^2\varphi}{\partial x\partial y}\left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x}\right)\right) d\Omega\right|\\ &= \left|\int_\Omega \left(\left(\frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi_n}{\partial y^2} + \frac{\partial^2\varphi_n}{\partial x^2}\right)\frac{\partial \delta u}{\partial x} - \left(\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi_n}{\partial x\partial y}\right)\right.\right.\\ &\qquad \left.\left.\times\left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x}\right)\right) d\Omega\right| \leqslant C_1\|\varphi - \varphi_n\|_H,\\ &C_1 = \left[\int_\Omega \left(4\left(\frac{\partial \delta u}{\partial x}\right)^2 + \left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x}\right)^2\right) d\Omega\right]^{1/2}.\end{aligned}$$

Thence according to (3.2), we obtain

$$\int_\Omega \left[\left(\frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial x^2}\right)\frac{\partial \delta u}{\partial x} - \frac{\partial^2\varphi}{\partial x\partial y}\left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x}\right)\right] d\Omega = 0. \tag{3.3}$$

Then

$$\int_\Omega \left[(\sigma_x - \sigma_y)\frac{\partial \delta u}{\partial x} + \tau\left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x}\right)\right] d\Omega - \int_{S_\sigma} [g(S)\,\delta v_n + h(S)\,\delta v_t]\, dS - \int_\Omega (f_x\delta u + f_y\delta v)\, d\Omega = 0. \tag{3.4}$$

follows from (3.1).

Thus the corresponding stress field (2.3) for an element $\varphi \in H$ satisfied the equilibrium equations (1.1) and the boundary conditions (1.2) in the generalized sense (3.4). In particular, if $\varphi \in M \subset H$, then Eq. (1.1) and the boundary conditions (1.2) are satisfied in the usual sense for the stresses (2.3).

We note in conclusion that the results obtained are valid for the entire region Ω occupied by the medium, independently of the distribution of rigid and plastic regions. A proof of the uniqueness of the stress field only for those parts of the body in which the deformation rates are different from zero is given in [2].

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PROBLEM OF PURE SHEAR OF A VISCOPLASTIC MEDIUM BETWEEN TWO NONCOAXIAL CIRCULAR CYLINDERS

A. V. Rezunov and A. D. Chernyshov

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The problem of the flow of a viscoplastic material between two noncoaxial circular cylinders is discussed. An approximate solution is found with the help of the iterative method described in [1, 2]. Analytic methods of solving similar problems are discussed in [3–4]. An approximate solution is found in [6, 7] with the use of iterative methods [8].

1. The problem is solved in a cylindrical coordinate system. The axis Oz is directed parallel to the generating lines of the cylinders, the contours of whose transverse cross section are specified by the equations $R_0 = R_0(\varphi)$ and $R_1 = R_1(\varphi)$. The outer cylinder is fixed, and the inner one moves in the positive direction of the axis Oz with velocity v_*. In this case only one velocity component $v_z = v(r, \varphi)$ is different from zero. In the flow under discussion the components of the deformation rate tensor are of the form

$$e_{rr} = e_{\varphi\varphi} = e_{zz} = e_{r\varphi} = 0, \quad e_{rz} = \frac{1}{2}\frac{\partial v}{\partial r}, \quad e_{\varphi z} = \frac{1}{2r}\frac{\partial v}{\partial \varphi}. \tag{1.1}$$

We will write the relation between the components of the stress tensor σ_{ij} and the components of the deformation rate tensor e_{ij} for a viscoplastic medium with the Miesz plasticity condition in the form [9]

$$\sigma_{ij} = \left(\frac{\sqrt{2}k}{\sqrt{e_{kl}e_{kl}}} + 2\mu\right)e_{ij} - p_1\delta_{ij}, \tag{1.2}$$

where p_1 is the hydrostatic pressure, k is the yield point, and μ is the viscosity coefficient. Substituting (1.1) into (1.2), we obtain

$$\sigma_{rr} = \sigma_{\varphi\varphi} = \sigma_{zz} = -p_1, \ \sigma_{r\varphi} = 0,$$
$$\sigma_{rz} = \frac{k + \mu\gamma}{\gamma}\frac{\partial v}{\partial r}, \quad \sigma_{\varphi z} = \frac{k + \mu\gamma}{r\gamma}\frac{\partial v}{\partial \varphi}, \quad \gamma = \sqrt{\left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v}{\partial \varphi}\right)^2}. \tag{1.3}$$

We write the equilibrium equations

$$\frac{\partial p_1}{\partial r} = \frac{\partial p_1}{\partial \varphi} = 0, \quad \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\varphi z}}{\partial \varphi} + \frac{\sigma_{rz}}{r} - \frac{\partial p_1}{\partial z} = 0. \tag{1.4}$$

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